Mixed Integer Linear Programming (MILPs)

Part III - Solving MILPs

Prof. Davide M. Raimondo

Acknowledgment: thanks to Andrea Pozzi for his help with these slides

Mixed Integer Linear Programming

A Mixed Integer Linear Programming (MILP) formulation implies the presence of **continuous** ($x \in \mathbb{R}_+^{n_x}$) and **discrete** $\left(y \in \mathbb{Z}_+^{n_y}\right)$ variables.

$$\boxed{\begin{aligned} z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\,\mathbf{y} \\ s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{y} \in \mathbb{Z}_+^{n_y} \end{aligned}}$$

Binary MILP

We consider in the following an MILP in which **the discrete variables are restricted to be binary** ($y \in \{0,1\}^{n_y}$).

$$\boxed{\begin{aligned} z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\,\mathbf{y} \\ s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{y} \in \mathbb{Z}_+^{n_y} \end{aligned}}$$

Important note: every MILP can be reformulated as a binary MILP!

Binary MILP Issues

$$\begin{aligned} z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\,\mathbf{y} \\ s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{y} \in \mathbb{Z}_+^{n_y} \end{aligned}$$

In the search for a rigorous solution of a MILP we face the following limitations:

- **NO local derivative** information **for integer variables** (no gradient methods)
- LP standard soultion methods, e.g. **simplex, ONLY when** the values for **all the discrete variables are** assumed to be **known.**

→ MILP solution is more difficult than LP (counter-intuitive)

Total enumeration approach: solve all the possible LPs (2^{n_y}) by fixing the values of the binary variables.

Optimal cost of the MILP: obtained by comparing the LPs optimal costs.

Optimal value of the optimization variables: binary values corresponding to the best LP and optimal value of the corresponding continuous variables.

Total Enumeration

$$
\begin{aligned}
z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\,\mathbf{y} \\
s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\
& \mathbf{x} \geq 0 \\
& \mathbf{y} \in \mathbb{Z}_+^{n_y}
\end{aligned}
$$

The total enumeration method is affected by an **exponential growth** of the computational burden with respect to the number of binary variables.

Total enumeration is affordable only for problems with few discrete variables

SOLUTION: Branch-and-Bound Search

Solving MILP: branch and bound

Construct a **sequence of simpler subproblems** whose solution converges to the original MILP one.

The subproblems are simpler than the original one since they are based on **relaxations.**

The **number of solved subproblems should be less (on average)**

than with the **total enumeration** approach (2^{n_y}).

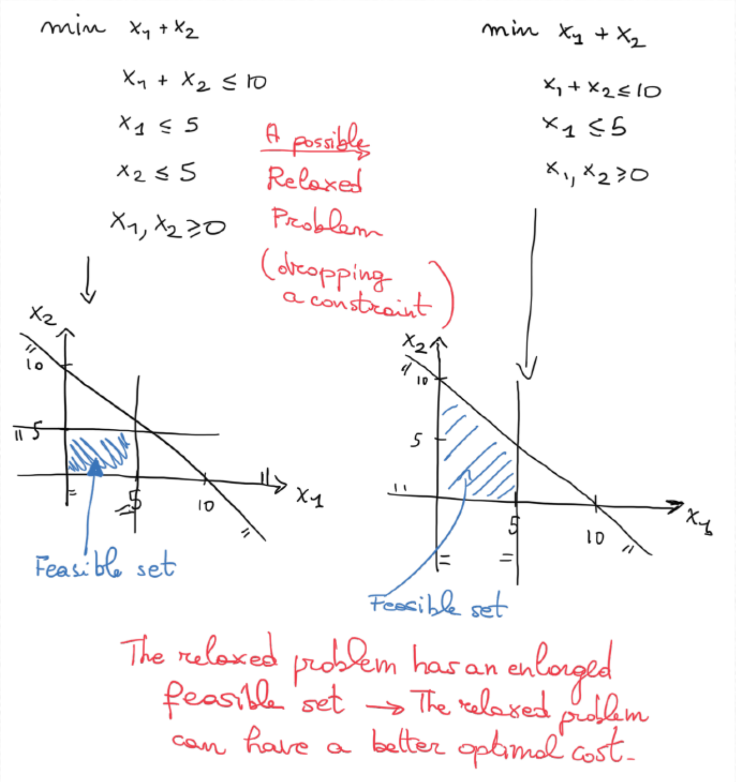
Relaxations of MILP

Definition:

A model C is a constraint relaxation of a model I if:

- every feasible solution in I is also feasible in C
- C and I have the same cost function

The relaxation can be done by dropping a constraint, relaxing the RHS of the inequality constraint, etc…

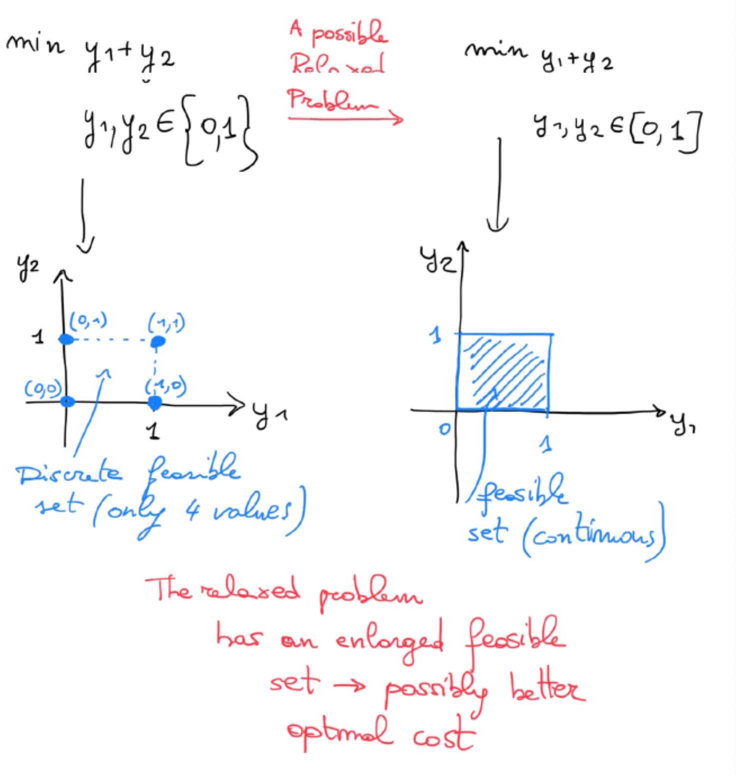


LP Relaxation

A MILP can be relaxed by **considering any discrete variables as continuous**, maintaining all the other constraints

$$\mathbf{y} \in \{0,\, 1\}^{n_y} \rightarrow \mathbf{y} \in [0,\, 1]^{n_y}$$

This allows us to exploit all the advantages of the methods available for the solution of LP problems.



Feasibility of MILP

Theorem

If the LP relaxation of a MILP is infeasible, then the original problem is infeasible

LP relaxation infeasible $\Rightarrow$ MILP infeasible

The viceversa is not true.

LP relaxation infeasible $\not\Leftarrow$ MILP infeasible

Lower and Upper Bounds

For a maximization MILP

The optimal value of the LP relaxation constitutes an **upper bound** for the original problem.

For a minimization MILP

The optimal value of the LP relaxation constitutes a **lower bound** for the original problem.

Optimal Solutions of MILP

Theorem

If the optimal solution of the LP relaxation is feasible for the original MILP, **then it is optimal also for the original problem.**

False intuition

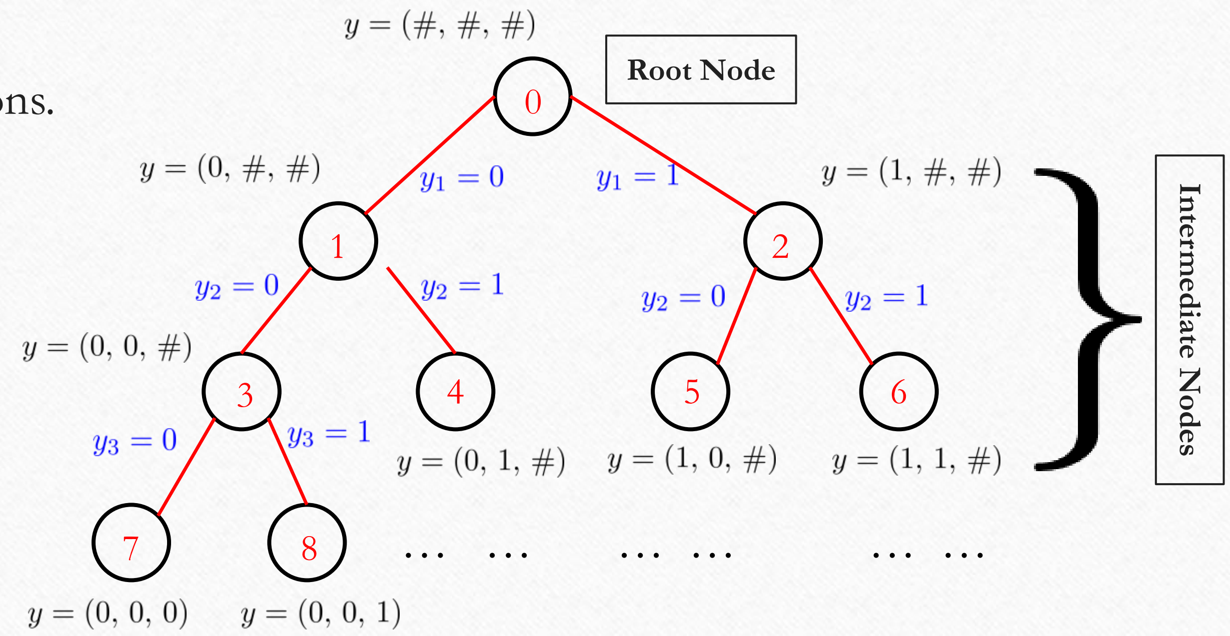
It is not true in general that an optimal solution for the MILP can be obtained by **rounding** the optimal solution of the LP relaxation. This can only provide a «good» feasible solution.

Branch and Bound Tree

Nodes of the tree represent the partial solutions.

Edges of the tree indicate which variable is fixed during the **branch** operation.

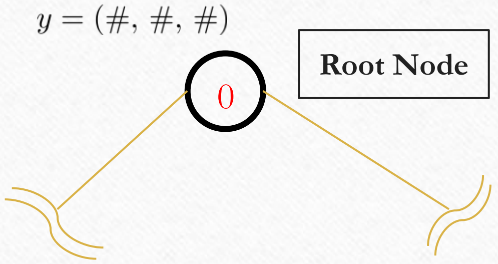
Root node: all the discrete variables are free.



Relaxation at Root Node

The solution of the LP relaxation at root node provides:

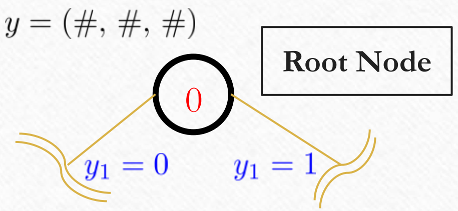
- lower bound for the global optimum of a minimization MILP
- upper bound for the global optimum of a maximization MILP



$$
\begin{aligned}
z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\ \mathbf{y} \\
s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\
& \mathbf{x} \geq 0 \\
& \mathbf{y} \in \{0, 1\}^{n_y}
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\ \mathbf{y} \\
s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\
& \mathbf{x} \geq 0 \\
& \mathbf{y} \in [0,\ 1]^{n_y}
\end{aligned}
$$

Relaxation at Root Node

The solution of the **LP relaxation at the root node** can provide:

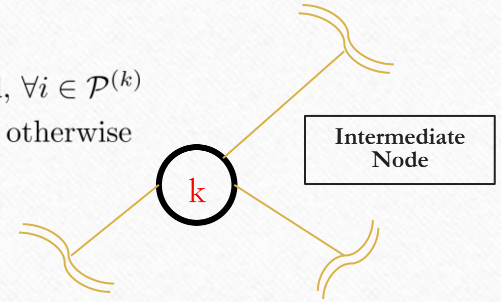


- no feasible solution: MILP infeasible (**terminate by infeasibility**)
- the optimal solution of the LP relaxation has binary value for the discrete variables (**terminate by completion**): we find the solution of the original MILP
- some discrete variables of the LP relaxation optimal solution are fractional values (branch):
  - **The tree needs to be further explored.** Choose one discrete variable and explore the two partial solution generated by fixing such variable to 0 and 1.

Intermediate Nodes

Starting from an MILP, the **candidate problem** associated with an **intermediate node** is the model obtained by fixing the discrete variables as in the considered partial solution.

$$y_i = \begin{cases} \text{fixed, } \forall i \in \mathcal{P}^{(k)} \\ \text{free, otherwise} \end{cases}$$



$$\begin{aligned} z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\ \mathbf{y} \\ s.t.\ & \mathbf{Ax} + \mathbf{Ey} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{y} \in \{0,1\}^{n_y} \end{aligned}$$

$\mathcal{P}^{(k)} \rightarrow$ fixed set

$$\begin{aligned} z_I^* = \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\ \mathbf{y} \\ s.t.\ & \mathbf{Ax} + \mathbf{Ey} = \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & y_i = \begin{cases} \text{fixed, } \forall i \in \mathcal{P}^{(k)} \\ \in \{0,\ 1\}, \text{ otherwise} \end{cases} \end{aligned}$$

LP Relaxation
of intermediate node

Incumbent Solution

While exploring the tree, we indicate with **incumbent solution**

the best feasible solution for the original MILP found so far.

The incumbent solution provides:

- **upper bound** for a minimization MILP
- **lower bound** for a maximization MILP

The incumbent solution may have been found during the search.

The B&B algorithm is efficient if not all nodes get explored. This happens when partial solutions are terminated at an early stage (meaning I do not explore the nodes below) by bounding with the LP relaxation solution and the incumbent one.

Terminating Partial Solution

- The candidate problem has an infeasible LP solution: - terminating by infeasibility **(no need to explore that branch any further!)**

- The optimal solution of the LP relaxation of the candidate problem is not better then the incumbent solution: no feasible completions of the candidate problem can improve the incumbent (terminating by value dominance). **Again, no need to explore that branch any further!**

- The LP relaxation of the candidate problem has an optimum with all the discrete variables binary
  1. terminate by completion the candidate problem
  2. update incumbent solution
  3. Do not explore the branch any further **(the optimum solution for that set of fixed binaries is already found)**

Terminating Branch and Bound

The search is over when all the partial solutions are branched or terminated. In this case, if an incumbent solution exists it is the global optimum, otherwise the problem is infeasible.

The algorithm can be stopped in advance if a certain threshold is achieved:

$$\frac{z_{ub} - z_{lb}}{\frac{1}{2}|z_{ub} + z_{lb}|} < \epsilon$$

where z_{lb} and z_{ub} are the lower and upper bounds for the optimal solution, given by the incumbent solution and the optimum of LP relaxations.

Branching Variables and Nodes Selection

The **choice of branching a certain discrete variable** can be done **heuristically**:

- consider for branching the discrete variables which have **fractional value** in the optimum of the candidate problem

Also **the choice of the node to be explored** can be done **heuristically**:

- Depth-first
- Best-first
- …